Homeomorphisms of Banach spaces over non-Archimedean fields with products of fields.

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Аннотация

The article is devoted to topological homeomorphisms of Banach spaces over complete non-Archimedean normed infinite fields with products of copies of the fields. [1]

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1 Introduction.

Mathematical analysis over infinite non-Archimedean normed fields is developing fast during recent years [12, 13, 14, 5]. Certainly structure of totally disconnected topological spaces, as well as manifolds and Banach spaces over non-Archimedean infinite fields is important not only for general topology, but also for mathematical analysis. Previously non-archimedean polyhedral expansions of totally disconnected

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$T_1 \cap T_{3.5}$ topological spaces and non-archimedean Banach manifolds were studied in [6, 7, 8, 9].

The Banach space c_0 in the non-archimedean analysis plays the same principal role as the Hilbert space l_2 in the classical analysis over $\mathbf{R}$ (see Theorems 5.13 and 5.16 in [12]). This article is devoted to topological homeomorphisms of Banach spaces over complete non-Archimedean normed infinite fields with products of copies of the fields (see Theorem 2 below). All main results of this paper are obtained for the first time.

2 Homeomorphisms of linear topological spaces.

1. Notation. For a topological space X supplied with a metric ρ defining its topology let $B(X, y, r)$ denote a ball $\{z \in X : \ \rho(z, y) \le r\}$ of radius $r > 0$ containing a marked point y. Suppose that $\mathbf{K}$ is a field with a multiplicative norm having values in $[0, \infty) \subset \mathbf{R}$ satisfying the inequality

(1) $|x + y| \le \max(|x|, |y|)$ for all elements $x, y \in \mathbf{K}$.

Such norm is called non-Archimedean, where $\Gamma_{\mathbf{K}} := \{|x| : \ x \in \mathbf{K}, x \ne 0\}$ is a multiplicative group. For example, $\mathbf{Q}_p \subset \mathbf{K}$ or $\mathbf{F}_p(\theta) \subset \mathbf{K}$, where $\mathbf{Q}_p$ denotes the field of p-adic numbers, $\mathbf{F}_p(\theta)$ denotes the field of formal Laurent series over the finite field $\mathbf{F}_p$, where $p > 1$ is a prime number. Each p-adic number $x \in \mathbf{Q}_p$ has the decomposition $x = \sum_{n=N}^{\infty} a_n p^n$, where $a_n \in \{0, 1, ..., p-1\}$ for each n, $N = N(x) \in \mathbf{Z}$ is an integer number so that $a_N \ne 0$, $|x| = |x|_p = p^{-cN}$, $c > 0$ is a constant on $\mathbf{Q}_p$, particularly, $c = 1$ can be taken, as usually $\mathbf{Z}$ denotes the ring of all integers. The characteristic of the p-adic field is zero $char(\mathbf{Q}_p) = 0$, while the characteristic of $\mathbf{F}_p(\theta)$ is $p = char(\mathbf{F}_p(\theta)) > 1$ positive.

Each element $x \in \mathbf{F}_p(\theta)$ has the series decomposition $x = \sum_{n=N}^{\infty} b_n \theta^n$, where $N = N(x) \in \mathbf{Z}$, $b_n \in \mathbf{F}_p$, $b_N \ne 0$, $|x| = |x|_p = p^{-cN}$, $c > 0$ is a constant on the field $\mathbf{F}_p(\theta)$, particularly, $c = 1$ can be taken. These fields differ by their multiplication and addition rules.

Then it is possible to take an algebraic or a transcendental extension of an initial field and its uniform or spherical completion if it is not such. The

field $\mathbf{C}_p$ of complex p-adic numbers is obtained as the uniform completion of a field containing all finite algebraic extensions of the p-adic field $\mathbf{Q}_p$ with the norm extending that of on $\mathbf{Q}_p$. The fields $\mathbf{Q}_p$ and $\mathbf{F}_p(\theta)$ are locally compact, the field $\mathbf{C}_p$ is not locally compact. The normalization group $\Gamma_{\mathbf{K}} := \{|x| : x \in \mathbf{K}, x \ne 0\}$ is multiplicative and commutative, for $\mathbf{K} = \mathbf{Q}_p$ it is isomorphic with $\{p^n : n \in \mathbf{Z}\}$, for $\mathbf{K} = \mathbf{C}_p$ it is isomorphic with $\{p^x : x \in \mathbf{Q}\}$, where $\mathbf{Q}$ denotes the field of all rational numbers. A larger field $\mathbf{U}_p$ exists so that $\mathbf{C}_p$ can be isometrically embedded into $\mathbf{U}_p$ and the normalization group $\Gamma_{\mathbf{U}_p}$ is isomorphic with $\{p^x : x \in \mathbf{R}\}$ (see [2, 13, 15]).

More generally a field $\mathbf{K}$ having a multiplicative norm $|x|$ with values in a linearly ordered commutative topological ring $\mathcal{R}$ can be considered so that $|x+y| \le \max(|x|, |y|)$. Suppose that the ring $\mathcal{R}$ is complete as the uniform space (see [3]) and 0 denotes the neutral element relative to the addition and 1 is the unit element relative to the multiplication in $\mathcal{R}$; moreover, if $p > 1$ in $\mathcal{R}$, then $\lim_{n\to+\infty} p^{-n} = 0$, where $n \in \mathbf{N}$. We consider the case, when an element $x \in \mathbf{K}$ exists so that $|x| = p > 1$. For example, $\mathcal{R} \subset \mathbf{R}^\gamma$, where γ is an ordinal, while elements z in $\mathbf{R}^\gamma$ are ordered lexicographically: $y < z$, if $y_j = z_j$ for each $j < k$ and $y_k < z_k$ for some $k \in \gamma$, where $\mathbf{R}$ denotes the field of real numbers. Non-zero elements $x \ne 0$ in the field $\mathbf{K}$ have norms belonging to the multiplicative group G of the ring $\mathcal{R}$. We suppose that the normalization group $\Gamma_{\mathbf{K}} := \{|x| : x \in \mathbf{K} \setminus \{0\}\}$ of the field $\mathbf{K}$ is infinite and the closure of $\Gamma_{\mathbf{K}}$ in the ring $\mathcal{R}$ contains the zero point 0, naturally, $\Gamma_{\mathbf{K}} \subset \{z \in \mathcal{R} : z > 0\}$; also $|x| = 0$ if and only if $x = 0 \in \mathbf{K}$.

Traditionally $c_0(\alpha, \mathbf{K}) =: c_0(\alpha)$ denotes the normed space over the field $\mathbf{K}$ consisting of all nets $x = \{x_j : j \in \alpha, x_j \in \mathbf{K}\}$ so that for each $\epsilon > 0$ the set $\lambda(x, \epsilon) := \{j \in \alpha : |x_j| > \epsilon\}$ is finite, where α is a set, the norm of x is:

(2) $\|x\| := \sup_{j\in\alpha} |x_j|$.

That is, either $|\alpha| < \aleph_0$ or $\Gamma_{\mathbf{K}}$ is discrete in $\mathbf{R}$ or $\mathcal{R}$ relative to the interval topology induced by its linear ordering.

For a normed $\mathbf{K}$-linear space E two vectors $x, y \in E$ are called orthogonal, if $\|ax + by\| = \max(\|ax\|, \|by\|)$ for all $a, b \in \mathbf{K}$. For a real number $0 < t \le 1$ a finite or an infinite sequence of elements $x_j \in E$ is called t-orthogonal, if $\|a_1x_1 + ... + a_mx_m + ...\| \ge t \max(\|a_1x_1\|, ..., \|a_mx_m\|, ...)$ for each $a_1, ..., a_m, ... \in$

$\mathbf{K}$ with $a_1x_1 + ... + a_mx_m + ... \in E$.

The standard orthonormal in the non-Archimedean sense base in $c_0(\alpha, \mathbf{K})$ is $e_j := (0, ..., 0, 1, 0, ...)$ with 1 at the j-th place. The space $c_0(\alpha, \mathbf{K})$ is Banach, when a non-Archimedean field $\mathbf{K}$ is complete as a uniform space. Henceforward, we consider the field $\mathbf{K}$ complete as the uniform space, if something other will not be specified.

Let ω_0 denote the first countable ordinal, for example, $\mathbf{N} := \{1, 2, 3, ...\}$.

We consider the product $\prod_{j\in\alpha} X_j$ of topological spaces X_j supplied with the Tychonoff (product) topology τ_{ty} with the base $U = \prod_{j\in\alpha} U_j$, where each U_j is open in X_j and only a finite number of U_j is different from X_j for a given U (see [3]). Henceforth, a locally compact field $\mathbf{K}$ is considered so that $\Gamma_{\mathbf{K}}$ is discrete. In this case let p be such that

$p^{-1} := \sup\{|x| : \ x \in \mathbf{K}, \ |x| < 1\}$.

2. Theorem. *The Banach space* $c_0(\omega_0, \mathbf{K}) =: c_0$ *over a uniformly complete non-Archimedean field* $\mathbf{K}$ *supplied with its norm topology* τ_n *is topologically homeomorphic with the countable product* $\mathbf{K}^{\omega_0}$ *of the non-Archimedean normed infinite field* $\mathbf{K}$*, where* $\mathbf{K}^{\omega_0}$ *is supplied with the Tychonoff topology* τ_{ty}*.*

The proof of this theorem is divided into several steps.

3. Remark. The condition that the field is infinite is essential. If the field is finite, then it is discrete and compact, consequently, the product $\mathbf{K}^{\omega_0}$ of compact topological spaces is compact. But the linear topological space $\mathbf{c}_0(\omega_0, \mathbf{K})$ is not compact, since the covering by clopen (closed and open simultaneously) balls $B(\mathbf{c}_0(\omega_0, \mathbf{K}), e_jx_m, 1/p)$, $j \in \omega_0$, is infinite and has not any finite sub-covering, where $\{x_m : m\}$ are all distinct elements of the field $\mathbf{K}$, $|x| = 1$ for each $x \neq 0$, $|0| = 0$, $p > 1$.

On the other hand, for the cardinality $card(\alpha) > \aleph_0 := card(\omega_0)$ of the set α greater than $\aleph_0$ the base of neighborhoods of zero in $\mathbf{K}^\alpha$ is uncountable, but $c_0(\alpha, \mathbf{K})$ has a countable base of neighborhoods of zero. Therefore, the topological spaces $c_0(\alpha, \mathbf{K})$ and $\mathbf{K}^\alpha$ are not homeomorphic when $card(\alpha) > \aleph_0$.

A topology on the product $\prod_{j\in\alpha} X_j$ of spaces stronger than the Tychonoff topology is given by the base $U = \prod_{j\in\alpha} U_j$, where each U_j is open in X_j. This topology is called the box topology τ_b [10].

We use the notation $\mathbf{s} := \mathbf{K}^{\omega_0}$, $\mathbf{s}^\alpha := \prod_{j\in\alpha} \mathbf{K}_j$ for a subset $\alpha \subset \omega_0$, where $\mathbf{s}$ and $\mathbf{s}^\alpha$ are supplied with the product Tychonoff topology.

4. Lemma. *If $\alpha \subset \omega_0$ and $\beta = \omega_0 \setminus \alpha$ are disjoint subsets in ω_0 and $\alpha \neq \emptyset$ is non-void, then*

(1) $\mathbf{s}$ *and* $\mathbf{s}^\alpha \times \mathbf{s}^\beta$ *are homeomorphic;*

(2) $c_0(\omega_0, \mathbf{K})$ *and* $c_0(\alpha, \mathbf{K}) \times c_0(\beta, \mathbf{K})$ *are homeomorphic.*

(3). *Moreover, if α is infinite, then $\mathbf{s}^\alpha$ is homeomorphic with $\mathbf{s}$, while $c_0(\alpha, \mathbf{K})$ is homeomorphic with $c_0(\omega_0, \mathbf{K})$.*

$(1, 3)$. This Lemma is evident, since $card(\omega_0^2) = card(\omega_0) = \aleph_0$.

(2). If $x \in c_0(\alpha, \mathbf{K})$ and $y \in c_0(\beta, \mathbf{K})$, then $\lim_j x_j = 0$ when α is infinite and $\lim_k y_k = 0$ when β is infinite, hence $\lim_l z_l = 0$, where $z_l = x_l$ for $l \in \alpha$, $z_l = y_l$ for $l \in \beta$, also $\alpha \cup \beta = \omega_0$. At the same time $\|z\| = \sup_{l\in\omega_0} |z_l| = \max(\|x\|, \|y\|)$.

5. Definitions. Let $\pi_\alpha : c_0(\omega_0, \mathbf{K}) \to c_0(\alpha, \mathbf{K})$ or $\pi_\alpha : \mathbf{s} \to \mathbf{s}^\alpha$ denote the natural projection. In particular, if $\alpha = \{j\}$ is a singleton we can write π_j instead of π_α.

A subset E in $c_0(\alpha, \mathbf{K})$ or in $\mathbf{s}^\alpha$ is called deficient in the j-th direction, if $\pi_j(E)$ is a singleton (i.e. consists of a single element). A subset E in $c_0(\alpha, \mathbf{K})$ or in $\mathbf{s}^\alpha$ is called infinitely deficient if for some infinite subset $\beta \subset \alpha$, each projection $\pi_j(E)$ is a singleton for every $j \in \beta$. In such case E will also be called deficient with respect to β. Henceforth, the term mapping will be used for continuous functions.

Suppose that A is a topological space and B is its subset and f is a mapping of B into A. One says that the mapping f is limited by an open covering W of A if for each point $x \in B$ there exists an element $V_x \in W$ so that $x \in V_x$ and $f(x) \in V_x$.

If A is a metric space supplied with a metric ρ, then the supremum $\sup_{V\in W} diam(V)$ is called the mesh of a covering W, where $diam(V) := \sup_{a,b\in V} \rho(a, b)$.

Let $f_1, f_2, ..., f_m, ...$ be a sequence of mappings such that the limit $\lim_{m\to\infty} f_m \circ f_{m-1} \circ ...f_1 : X \to X$ exists, where X is a topological space. This limit is denoted by $L\prod_{j=1}^\infty f_j$ and is called the infinite left product of the mappings f_j.

We consider the subsets $A_0 := \{x \in c_0 : \ \sup_{i\in \mathbf{N}} |\sum_{j=1}^i x_j| = 1\}$ and $E^j := \{x \in c_0 : \ x = (x_1, x_2, ...), \ x_k = 0 \ \forall k > j\}$ in $c_0 := c_0(\omega_0, \mathbf{K})$.

Lemma 6. *The topological spaces* $\mathbf{K}$ *and* $B(\mathbf{K}, 0, 1) \setminus \{1\}$ *are topologically homeomorphic, where* $\mathbf{K}$ *is a field (see §1).*

Proof. We take any element $p \in G$ so that $p > 1$ and $p = |x|$ for some invertible element $x \in \mathbf{K}$ (see §1). Therefore, $1/p^n = p^{-n}$ tends to zero in the topological ring $\mathcal{R}$ while $n \in \mathbf{N}$ tends to $+\infty$. Then the topological field $\mathbf{K}$ can be written as the disjoint union of clopen subsets $B(\mathbf{K}, 0, 1)$, $B(\mathbf{K}, 0, p) \setminus B(\mathbf{K}, 0, 1)$,...,$B(\mathbf{K}, 0, p^{n+1}) \setminus B(\mathbf{K}, 0, p^n)$, ... with $n \in \mathbf{N}$.

The norm in $\mathbf{K}$ is multiplicative, consequently, $B(\mathbf{K}, 0, p^n) = x^n B(\mathbf{K}, 0, 1)$ for each $n \in \mathbf{Z}$, where $XY := \{z = xy : \ x \in X, y \in Y\}$ for two subsets X and Y in $\mathbf{K}$. Thus subsets $B(\mathbf{K}, 0, p^{n+1})$ and $B(\mathbf{K}, 0, p^m)$ are isomorphic for all $n, m \in \mathbf{Z}$. Moreover, we have the equalities $B(\mathbf{K}, 0, 1) \setminus \{1\} = \bigcup_{n=0}^{\infty} [B(\mathbf{K}, 1, p^{-n}) \setminus B(\mathbf{K}, 1, p^{-n-1})]$ and $B(\mathbf{K}, y, r) = y + B(\mathbf{K}, 0, r)$ for each $y \in \mathbf{K}$ and $r > 0$. Each set $B(\mathbf{K}, 1, p^{-n}) \setminus B(\mathbf{K}, 1, p^{-n-1})$ or $B(\mathbf{K}, 0, p^{n+1}) \setminus B(\mathbf{K}, 0, p^n)$ is the disjoint union of balls $B(\mathbf{K}, y_j, p^m)$ or $B(\mathbf{K}, z_j, p^m)$ with $m = -n - 1$ or $m = n$ respectively.

On the other hand, the quotient ring $B(\mathbf{K}, 0, 1)/B(\mathbf{K}, 0, 1/p)$ exists. Its additive group is isomorphic with

$$[x^n B(\mathbf{K}, 0, 1)]/[x^n B(\mathbf{K}, 0, 1/p)] = B(\mathbf{K}, 0, p^n)/B(\mathbf{K}, 0, p^{n-1})$$
$$= x^n [B(\mathbf{K}, 0, 1)/B(\mathbf{K}, 0, 1/p)]$$

for each $n \in \mathbf{Z}$, where $x \in \mathbf{K}$ with $|x| = p$. Thus $\mathbf{K}$ and $B(\mathbf{K}, 0, 1) \setminus \{1\}$ can be presented as disjoint unions $\mathbf{K} = \bigcup_{\lambda\in\Lambda_1} B(\mathbf{K}, z_\lambda, r_\lambda)$ and $B(\mathbf{K}, 0, 1) \setminus \{1\} = \bigcup_{\mu\in\Lambda_2} B(\mathbf{K}, y_\mu, r_\mu)$ with $card(\Lambda_1) = card(\Lambda_2) \geq \aleph_0$. We take any mapping $\phi : \mathbf{K} \to B(\mathbf{K}, 0, 1) \setminus \{1\}$ such that $\phi : B(\mathbf{K}, z_\lambda, r_\lambda) \to B(\mathbf{K}, y_{\psi(\lambda)}, r_{\psi(\lambda)})$ is a homeomorphism. For example, ϕ can be chosen affine $x \mapsto a + bx$ on each ball $B(\mathbf{K}, z_\lambda, r_\lambda)$, where $\psi : \Lambda_1 \to \Lambda_2$ is a bijective surjective mapping. Thus $\phi : \mathbf{K} \to B(\mathbf{K}, 0, 1) \setminus \{1\}$ is the topological homeomorphism.

7. Remark. Using the preceding lemma we henceforth consider $\mathbf{s}$ as homeomorphic with

$$(1) \quad \mathbf{s} \simeq s = \prod_{j=1}^{\infty} [B(\mathbf{K}, 0, 1)_j \setminus \{1\}]$$

supplied with the Tychonoff product topology if **s** from §3 will not specified, where $B(\mathbf{K},0,1)_j = B(\mathbf{K},0,1)$ for each j. Then we put

$$(2)\quad s_* = \{y \in s: \lim_{m\to\infty}(1-y_m)...(1-y_1) = 0\} \setminus \bigcup_{n=1}^{\infty} \mathsf{E}^n,$$

where $\mathsf{E}^n := \{x \in s: x = (x_1, x_2, ...),\ x_k = 0\ \forall k > n\}$.

8. Lemma. *A ring or a field* $\mathbf{K}$ *has the natural uniformity and its completion* $\tilde{\mathbf{K}}$ *relative to this uniformity is a topological ring or a field correspondingly.*

Proof. The norm $|*|$ in $\mathbf{K}$ is multiplicative with values in $G \cup \{0\} \subset \mathcal{R}$. One can take a diagonal $\Delta := \{(x,y) \in \mathbf{K}^2 : x = y\}$ in the Cartesian product $\mathbf{K}^2 = \mathbf{K} \times \mathbf{K}$. This norm induces entourages of the diagonal in $\mathbf{K}^2$: $V_z := \{(x,y) \in \mathbf{K}^2 : |x-y| \le z\}$ for each $z \in G$. Therefore,

$(E1)$ $\bigcap_{z\in G} V_z = \Delta$,

since $x = y$ if and only if $|x-y| = 0$.

$(E2)$. If $z_1 < z_2$ then $V_{z_1} \subset V_{z_2}$, since $|x-y| < z_1$ implies $|x-y| < z_2$. Then we have also

$(E3)$ if $(x,y) \in V_z$ and $(y,\xi) \in V_b$, then $(x,\xi) \in V_{\max(z,b)}$, since $|x-\xi| \le \max(|x-y|, |y-\xi|)$.

Naturally, the inclusion $(x,y) \in V_z$ is equivalent to $(y,x) \in V_z$, since $|x-y| = |y-x|$. The family $\mathcal{E}$ of all entourages of the diagonal D in $\mathbf{K}$ provides the uniformity $\mathcal{U}$ in $\mathbf{K}$ compatible with its topology (see Chapter 8 [3]). The completion $\tilde{\mathbf{K}}$ relative to this uniformity $\mathcal{U}$ is the uniformly complete field, since the addition and multiplication operations are uniformly continuous on the ring, also the inversion operation on $\mathbf{K} \setminus \{0\}$ for the field and they have uniformly continuous extensions on either $\tilde{\mathbf{K}}$ or on $\tilde{\mathbf{K}} \setminus \{0\}$ respectively.

9. Notation. Let $\mathbf{K}$ be a uniformly complete non-Archimedean field. We define the subset

(1) $A_1 := \{x \in c_0 : \sup_{k\in\mathbf{N}} |\sum_{j=1}^k x_j| = 1, |1-\sum_{j=1}^{k+1} x_j| \le |1-\sum_{j=1}^k x_j|\ \forall k,\ \sum_{j=1}^\infty x_j = 1\}$ in c_0, also

(2) $A_1^* := A_1 \setminus \bigcup_{n=1}^\infty E^n$ (see §5).

Another larger subsets we define by the formula:

(3) $A_2 := \{x \in c_0 : \sup_{k\in\mathbf{N}} |\sum_{j=1}^k x_j| = 1, \sum_{j=1}^k x_j \ne 1\ \forall k,\ \sum_{j=1}^\infty x_j = 1\}$ in c_0, also

(4) $A_2^* := A_2 \setminus \bigcup_{n=1}^\infty E^n$.

Let A_1 and A_1^* and also A_2 and A_2^* be supplied with the topology inherited from the normed space c_0.

10. Lemma. *The topological spaces A_1^* and s_* are homeomorphic.*

Proof. We define the following mapping $q : A_1^* \to s$ so that $q(x) = y$, where $x = (x_1, x_2, ...) \in A_1^*$, $y = (y_1, y_2, ...) \in s$ (see §7). The domain of y_j is $B(\mathbf{K}, 0, 1) \setminus \{1\}$ for each $j \in \mathbf{N}$.

If $x \in A_1$ and $|1 - \sum_{j=1}^k x_j| = 0$, then $|1 - \sum_{j=1}^m x_j| \le |1 - \sum_{j=1}^k x_j| = 0$ for all $m > k$, consequently, $1 \ne \sum_{j=1}^k x_j$ for each $x \in A_1^*$ and $k \in \mathbf{N}$, since $|1 - b| = 0$ is equivalent to $b = 1$ and x does not belong to $\bigcup_n E^n$.

We take an arbitrary vector $x = (x_1, x_2, ...)$ in A_1^*. Since $x_1 \in B(\mathbf{K}, 0, 1) \setminus \{1\}$, we can put $y_1 = x_1$. Moreover, we have

(1) $|1 - x_1 - ... - x_k| \le \max(|1 - x_1 - ... - x_{k+1}|, |x_{k+1}|)$ and

(2) $|x_{k+1}| \le \max(|1 - x_1 - ... - x_{k+1}|, |1 - x_1 - ... - x_k|) = |1 - x_1 - ... - x_k|$ for each k and each $x \in A_1$. So we can take $y_2 = 1 - (1 - x_1 - x_2)/(1 - x_1) = x_2/(1 - x_1)$. By induction if $x_1, ..., x_m$ are marked, then $x_{m+1} \in B(\mathbf{K}, -x_1 - ... - x_m, 1) \setminus \{1\}$ and it is sufficient to put

(3) $y_{m+1} = 1 - (1 - x_1 - ... - x_{m+1})/(1 - x_1 - ... - x_m) = x_{m+1}/(1 - x_1 - ... - x_m)$, consequently, $y_j \in B(\mathbf{K}, 0, 1)$ for each $j \in \mathbf{N}$. Therefore,

(4) $(1 - x_1 - ... - x_{m+1}) = (1 - y_{m+1})(1 - x_1 - ... - x_m) = (1 - y_{m+1})...(1 - y_1)$ for each natural number $m \in \mathbf{N} = \{1, 2, 3, ...\}$. The inverse mapping q^{-1} is given by:

(5) $x_1 = y_1$, $x_2 = y_2(1 - y_1)$, $x_{m+1} = y_{m+1}(1 - x_1 - ... - x_m) = y_{m+1}(1 - y_m)...(1 - y_1)$ for every natural number $m \in \mathbf{N}$, consequently, $|x_j| \le 1$ for each j and $|\sum_{j=1}^k x_j| \le \max_{j=1}^k |x_j| \le 1$ for each $k \in \mathbf{N}$. Thus $x_{m+1} = 0$ is equivalent to $y_{m+1} = 0$, since $y_j \ne 1$ for each $y \in s$. But for each $y \in s_*$ the set $\{j : \ y_j \ne 0\}$ is infinite, which is equivalent to the fact that the set $\{j : \ x_j \ne 0\}$ is infinite for $x = q^{-1}(y)$. That is, $q^{-1}(s_*) \subset A_1^*$.

From the definition of q one can lightly see that $\lim_{k\to\infty} \sum_{j=1}^k x_j = 1$ is equivalent to the fact that the limit

(6) $\lim_{j\to\infty}(1 - y_j)...(1 - y_1) = 0 \in B(\mathbf{K}, 0, 1)$ exists. Thus the mapping q is bijective from A_1^* onto s_*, moreover, q and its inverse $g = q^{-1} : s_* \to A_1^*$ are coordinate-wise continuous.

Let x^n be a converging sequence in A_1^*, $\lim_{n\to\infty} x^n = x \in A_1^*$, then

$y_j^{n+m} - y_j^n = x_j^{n+m}/(1 - x_1^{n+m} - ... - x_{j-1}^{n+m}) - x_j^n/(1 - x_1^n - ... - x_{j-1}^n)$ and

$|y_j^{n+m} - y_j^n| \le$

$\max_{0\le i\le j-1} |x_j^{n+m} x_i^n - x_j^n x_i^{n+m}|/[|1 - x_1^{n+m} - ... - x_{j-1}^{n+m}||1 - x_1^n - ... - x_{j-1}^n|] \le \max_{0\le i\le j-1}(|x_j^{n+m} - x_j^n||x_i|^n, |x_j^n||x_i^{n+m} - x_i^n|)/[|1 - x_1^{n+m} - ... - x_{j-1}^{n+m}||1 - x_1^n - ... - x_{j-1}^n|]$,

where $x_0^n = 1$, consequently, the mapping $q : A_1^* \to s_*$ is continuous, since s_* is in the topology inherited from the Tychonoff product topology on s (see §7).

If y^n is a converging sequence in s_*, then

$x_j^{n+m} - x_j^n = y_j^{n+m}(y_{j-1}^{n+m} - 1)...(y_1^{n+m} - 1) - y_j^n(y_{j-1}^n - 1)...(y_1^n - 1)$, consequently,

(7) $|x_j^{n+m} - x_j^n| \le \max(|y_j^{n+m} - y_j^n||(y_{j-1}^{n+m} - 1)...(y_1^{n+m} - 1)|,$

$|y_j^n||(y_{j-1}^{n+m} - 1)...(y_1^{n+m} - 1) - (y_{j-1}^n - 1)...(y_1^n - 1)|)$ and

$\|x^{n+m} - x^n\| = \sup_j |x_j^{n+m} - x_j^n|$.

But $|x_j| \le 1$ for each j and $\lim_{j\to\infty} x_j = 0$. For each $\epsilon > 0$ there exists a natural number $j_0 > 0$ such that $|x_j| < \epsilon$ for each $j > j_0$. Then for each $\delta > 0$ there exists a natural number n_0 such that $|y_k^n - y_k| < \delta$ for each $k = 1, ..., j_0$. Choose $\delta > 0$ such that $|x_k - x_k^n| < \epsilon$ for each $k = 1, ..., j_0$ and $n > n_0$. Then

$|x_j| \le \max(|1 - x_1 - ... - x_j|, |1 - x_1 - ... - x_{j-1}|) \le |1 - x_1 - ... - x_{j_0}|$

$= |(1 - y_{j_0})...(1 - y_1)|$ and

(8) $|x_j^n| \le |(1 - y_{j_0}^n)...(1 - y_1^n)|$

for all $n \in \mathbf{N}$ and each $j > j_0$.

Therefore, from Formulas $(7, 8)$ it follows that the mapping $g = q^{-1} : s_* \to A_1^*$ is continuous, since $(1 - y_k) \in B(\mathbf{K}, 0, 1)$ for each k, $|ab| \le |a||b|$ for all $a, b \in \mathbf{K}$, while

$|x_j - x_j^n| \le \max(|x_j|, |x_j^n|) \le |(1 - y_{j_0})...(1 - y_1)|(1 + \delta)$

for each $n > n_0$ and $j > j_0$, where $\delta > 0$ can be taken less than ϵ.

11. Lemma. *Each element $x \in A_2$ can be presented as $x = x^1 + x^2$, where $1 - x^1$ and $x^2 \in A_1$.*

Proof. Consider the bounded sequence

$a_n := |1 - \sum_{j=1}^k x_j| \le \max(1, |\sum_{j=1}^k x_j|) = 1$

in $\mathcal{R}$ and compose the new sequence $w_1 = a_1$, $w_n = a_n - a_{n-1}$ for each

$2 \le n \in \mathbf{N}$, consequently, $a_n = w_1 + ... + w_n$ for each $2 \le n \in \mathbf{N}$. Then we put $b_n := \max(w_n, 0)$ and $c_n := -\min(w_n, 0)$, consequently, $w_m = b_m - c_m$ and $0 \le b_m$ and $0 \le c_m$ for each $m \in \mathbf{N}$. Since $\sum_{j=1}^k x_j \ne 1$ $\forall k$, and $\sum_{j=1}^\infty x_j = 1$, we certainly have the conditions $a_n > 0$ for each $n \in \mathbf{N}$ and $\lim_n a_n = 0$.

Put now $d_n := \sup_{m\ge n}[\max(b_m, c_m) + a_{m-1}]$ and $e_n := \inf_{1\le m\le n} a_m$, where $a_0 := 0$, consequently, $0 \le d_{n+1} \le d_n \le 2$ and $0 \le e_{n+1} \le e_n \le 1$ for each $n \in \mathbf{N}$ and $\lim_n d_n = 0$ and $\lim_n e_n = 0$, since

$$||1 - \sum_1^{m-1} x_j| - |1 - \sum_1^m x_j|| \le |x_m| \le 1$$

for each $m \in \mathbf{N}$ and $x \in A_2$. From $a_n \le \max(b_n + a_{n-1},\ c_n + a_{n-1})$ one gets the inequality $a_n \le d_n$ for each natural number n. On the other hand, $e_n \le a_n$ and $a_n, e_n \in \Gamma_{\mathbf{K}} \bigcup \{0\}$ for each n.

If $y, z \in \mathbf{K}$, $|y| = r_1 \le r_2$, $|z| = r_2$, then $(r_2 - r_1) \le |y + z| \le r_2$.

Consider subsets $\{(\beta_n, \gamma_n) \in \mathbf{K}^2 :\ |\beta_n| = a_n,\ |1-\gamma_n| = e_n,\ |1-\beta_n-\gamma_n| = a_n\}$. Therefore, $1-\beta_n = x_1^1 + ... + x_n^1$ and $\gamma_n = x_1^2 + ... + x_n^2$ give two elements $1 - x^1$ and $x^2 \in A_1$ such that $x = x^1 + x^2$, where $x^l = (x_1^l, x_2^l, ...)$, $x_k^l \in \mathbf{K}$ for each $k \in \mathbf{N}$ and $l = 1, 2$.

11.1. Corollary. *There are embeddings:*

(1) $A_2 \hookrightarrow (1 - A_1) \cup A_1 \hookrightarrow c_0$ *and*

(2) $A_2^* \hookrightarrow (1 - A_1^*) \oplus A_1^* \hookrightarrow c_0$.

Proof. The first embedding follows from Lemma 10. On the other hand, $(1 - A_1^*) \cap A_1^* = \emptyset$, since $\sum_{j=1}^k x_j \ne 1$ for each $x \in A_1^*$ and $k \in \mathbf{N}$ while $\sum_{j=1}^\infty x_j = 1$, but $\sum_{j=1}^\infty y_j = 0$ for each $y \in 1 - A_1^*$. Therefore, $(1 - A_1^*) \cup A_1^* = (1 - A_1^*) \oplus A_1^*$, since the mapping $x \mapsto \sum_{j=1}^\infty x_j$ is continuous from c_0 into $\mathbf{K}$, at the same time

$$|\sum_{j=1}^\infty x_j - \sum_{j=1}^\infty z_j| \le \sup_{j\in\mathbf{N}} |x_j - z_j| = \|x - z\|$$

for every $x, z \in c_0$.

12. Lemma. *The topological spaces A_1^* and A_2^* are homeomorphic.*

Proof. The Banach spaces c_0 and c are linearly topologically isomorphic, since $c_0 \oplus \mathbf{K}$ is linearly topologically isomorphic with c_0 and with c as well,

where $c = c(\omega_0, \mathbf{K})$,

$$(1) \quad c(\alpha, \mathbf{K}) := \{x : \ x = (x_j : j \in \alpha), \forall j \in \alpha \ \ x_j \in \mathbf{K},$$

$$\|x\| = \sup_{j \in \alpha} |x_j|, \exists \lim_j x_j \in \mathbf{K}\}.$$

If $y \in c_0$ put

(2) $x_1 = y_1$, $x_j = y_1 + ... + y_j$ for each $2 \le j \in \omega_0$.

Since $\lim_j y_j = 0$ and $|\sum_{j=n}^m y_j| \le \max_{n \le j \le m} |y_j|$ for each $1 \le n \le m$, the series $\sum_{j=1}^\infty y_j$ converges and $x \in c$. Consider A_1^* and A_2^* embedded into the Banach space c, $A_l^* \ni y \mapsto x = x(y) \in c$ (see Formula (2)). Take $x \in A_l^*$, then $\|x\| = \epsilon > 0$, where $l = 1, 2$. There exists $\xi \in A_l^*$ with $\|x\| = \|\xi\|$ and $\inf_j |\xi_j| = \delta > 0$ choosing $0 < \delta \le \min(1/p, \epsilon)$. Therefore, if $\xi \in A_1^*$, then $B(A_2^*, \xi, \delta/p) \subset A_1^*$. Indeed, if $|a - x_j| \le \delta/p$ and $|b - x_{j+1}| \le \delta/p$, then $|a| = |x_j|$ and $|b| = |x_{j+1}|$, where $a, b \in \mathbf{K}$. That is from $|x_{j+1}| \le |x_j|$ it follows, that $|b| \le |a|$.

Let $\omega(\mathbf{K})$ be the topological weight of the field $\mathbf{K}$, then A_1^* and A_2^* have coverings by disjoint families $\mathcal{F}_1$ and $\mathcal{F}_2$ of balls such as $B(A_2^*, \xi, r)$ with $0 < r \le \delta/p$ as above, where $|\mathcal{F}_1| = |\mathcal{F}_2| = \omega(\mathbf{K})|\Gamma_{\mathbf{K}}|\aleph_0$, as usually $|E| = card(E)$ denotes the cardinality of a set E. Each two balls $B(A_2^*, \xi, r_1)$ and $B(A_2^*, \eta, r_2)$ in A_2^* of radius less than $1/p$ are homeomorphic, since they are disjoint unions of $\omega(\mathbf{K})\aleph_0$ balls $B(A_2^*, z, r_3)$ of radius $r_3 = \min(r_1, r_2)/p$. Hence A_1^* and A_2^* are homeomorphic due to Corollary 11.1.

13. Definitions. A topological space X is called ultrametric, if its topology is given by an ultrametric ρ having values in $\Gamma_{\mathbf{K}} \cup \{0\}$ such that

(1) $\rho(x, y) \ge 0$ for every $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$; and a metric satisfies the ultrametric inequality which is stronger than the usual triangle inequality:

(3) $\rho(x, z) \le \max(\rho(x, y), \rho(y, z))$ for every $x, y, z \in X$.

If X is an ultrametric space and $H_t : X \times B(\mathbf{K}, 0, 1) \to X$ is a simultaneously continuous mapping so that $t \in B(\mathbf{K}, 0, 1)$ and $H_0 = id$, $H_t : X \to X$ is a homeomorphism for each t, then H_t is called an isotopy. The isotopy is called invertible, if H_t^{-1} is jointly continuous in $(x, t) \in X \times B(\mathbf{K}, 0, 1)$.

14. Lemma. *Let (X, ρ) be an ultrametric space, and let A and B be two non intersecting subsets in X. Suppose that $\mathbf{K}$ is an infinite non discrete*

non-archimedean field. Then there exists a continuous function $f : X \to \mathbf{K}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. Since A and B are closed subsets in X and $A \cap B = \emptyset$, then $\inf_{x\in A,\ y\in B} \rho(x, y) = d > 0$. A field $\mathbf{K}$ is non discrete, that is 0 is a limit point of a set $\{|x| : x \ne 0,\ x \in \mathbf{K}\}$ in a ring $\mathcal{R}$. Therefore, an element $r \in \Gamma_{\mathbf{K}}$ exists such that $0 < r < d$. We take two subsets $U := \{x \in X :\ \rho(B, x) \le r\}$ and $V := \{x \in X :\ \rho(A, x) \le r\}$, where $\rho(B, x) := \inf_{y\in B} \rho(x, y)$. From the ultrametric inequality it follows that two subsets U and V do not intersect, $U \cap V = \emptyset$. The set U is clopen in X, since $B(X, y, r) \subset U$ for each $y \in U$, where $B(X, y, r) := \{z \in X :\ \rho(y, z) \le r\}$, consequently, $X \setminus U$ is clopen in X. On the other hand, $A \subset V \subset (X\setminus U)$. Next we take a function $f : X \to \mathbf{K}$ such that $f(x) = 1$ for each $x \in U$ and $f(x) = 0$ for each $x \in X \setminus U$. This function f is continuous

15. Lemma. *Let $A_1, ..., A_n$ be closed non intersecting (i.e. disjoint) subsets in an ultrametric space (X, ρ). Suppose that $\mathbf{K}$ is an infinite non discrete non-archimedean field and $b_k \ne b_j \in \mathbf{K}$ for each $k \ne j$, $j, k = 1, ..., n$. Then there exists a continuous function $f : X \to \mathbf{K}$ such that $f(A_k) = \{b_k\}$.*

Proof. Subsets $A_1, .., A_n$ are closed and disjoint, consequently,

$$\min_{1\le j\ne k\le n} \inf_{x\in A,\ y\in B} \rho(x, y) = d > 0.$$

We choose $r \in \Gamma_{\mathbf{K}}$ such that $0 < r < d$ and take clopen subsets $U_j = \{x \in X :\ \rho(A_j, x) \le r\}$ for each $j = 1, ..., n$. Then $A_j \subset U_j$ for each j and $A_n \subset X \setminus (\bigcup_{k=1}^{n-1} U_j)$. These clopen subsets U_j are pairwise disjoint due to the ultrametric inequality. The field $\mathbf{K}$ is infinite, hence there are distinct elements $b_1, ..., b_n$, that is $b_k \ne b_j \in \mathbf{K}$ for each $k \ne j$, $j, k = 1, ..., n$. Take any distinct n elements $b_1, ..., b_n$ in $\mathbf{K}$. We construct a function $f : X \to \mathbf{K}$ such that $f(x) = b_j$ for each $x \in U_j$ with $j = 1, ..., n - 1$ and $f(x) = b_n$ for each $x \in X \setminus (\bigcup_{k=1}^{n-1} U_j)$. Since subsets U_j for each j and $X \setminus (\bigcup_{k=1}^{n-1} U_j)$ are clopen in X, then the function f is continuous.

16. Theorem. *Let (X, ρ) be an ultrametric space. Suppose that $\mathbf{K}$ is a non-archimedean infinite non discrete locally compact field. Let also M be a closed subset in X and let $f : M \to \mathbf{K}$ be a continuous function. Then there exists a continuous extension g of f on X, $g : X \to \mathbf{K}$.*

Proof. In view of Lemma 6 $\mathbf{K}$ and $B(\mathbf{K},0,1,)\setminus\{1\}$ are homeomorphic. Therefore, it is sufficient to consider continuous mappings $f : M \to B(\mathbf{K},0,1)$. In this case a constant $c \in \Gamma_{\mathbf{K}}$ exists such that $|f(x)| \le c$ for each $x \in M$. On the other hand, each locally compact field $\mathbf{K}$ has a discrete group $\Gamma_{\mathbf{K}}$.

We construct a sequence of functions $g_n : X \to \mathbf{K}$ a limit of which $g : X \to \mathbf{K}$ will be a continuous extension of f. Consider a continuous function $g_n|_M : M \to \mathbf{K}$ from a closed subset M in X. A field $\mathbf{K}$ is locally compact, consequently, the quotient ring $B(\mathbf{K},0,c)/B(\mathbf{K},0,c/p^n)$ is finite. Hence there are points $x_{n,j} \in B(\mathbf{K},0,c)$ such that $B(\mathbf{K},0,c)$ is a disjoint union of clopen balls $B(\mathbf{K},x_{n,j},c/p^n)$, where $j = 1,...,m(n) \in \mathbf{N}$. We take the clopen subsets $A_{n,j} := (g_n|_M)^{-1}(B(\mathbf{K},x_{n,j},c/p^n))$ in M and consider clopen subsets $U_{n,j} := \{x \in X : \ \rho(x,A_{n,j}) \le c/p^{n+1}\}$ in X. Thus $\bigcup_{j=1}^{m(n)} A_{n,j} = M$.

In view of Lemma 15 a continuous function $u_n : X \to \mathbf{K}$ exists satisfying the conditions

(1) $u_n(U_{n,j}) \subset B(\mathbf{K},x_{n,j},c/p^n)$ for each $j = 1,...,m(n)$ and

(2) $|u_n(x)| \le c$ for each point x in X. Therefore,

(3) $|u_n(x) - g_n(x)| \le c/p^n$ for each $x \in M$ and

(4) $M \subset \bigcup_{j=1}^{m(n)} U_{n,j} =: P_{n+1}$.

Each subset P_n is clopen in X and it is possible to take any continuous function $h : X \setminus P_2 \to \mathbf{K}$. We continue this process by induction from $n = 1$ with $g_1|_M = f$, further putting

(5) $g_{n+1}|_M = f : M \to \mathbf{K}$ and

(6) $g_{n+1}(y) = u_n(y)$ for each $y \in X \setminus (P_n \setminus P_{n+1})$ when $n \ge 2$ and

(7) $g_{n+1}(y) = h(y)$ for each $y \in X \setminus P_2$.

Then $P_{n+2} \subset P_{n+1}$ and P_{n+2} is clopen in P_{n+1} for each $n \in \mathbf{N} = \{1,2,3,...\}$ and

(8) $\bigcap_{n=2}^{\infty} P_n = M$.

Therefore, $g_{n+k} = g_{n+1}$ on $X \setminus P_{n+2}$ for each $k \ge 2$. From the construction above and formulas $(1-8)$ it follows, that $\lim_n g_n(x) = f(x)$ for each $x \in M$ and $g = \lim_n g_n$ is continuous from X into $B(\mathbf{K},0,c)$, since $|g_n(x) - g_{n+1}(x)| \le cp^{-n}$ for each $x \in X$ and $n \in \mathbf{N}$.

17. Example. Let f_j be a continuous surjective mapping from $B(\mathbf{K},0,1)$ into $B(\mathbf{K},0,1) \setminus B(\mathbf{K},0,1/p^j)$ such that $f_j(B(\mathbf{K},0,1) \setminus B(\mathbf{K},0,1/p^j)) = \{1\}$

and $f_j(B(\mathbf{K},0,1/p^{j+1}))=\{1\}$ and $f_j(B(\mathbf{K},0,1/p^j)\setminus B(\mathbf{K},0,1/p^{j+1}))=\{b_j\}$, $b_j\ne 0$ with $\lim_j b_j=0$. Next a mapping $H_t(x)=(x_1f_1(t),x_2f_2(t),...)$ exists for each $x=(x_1,x_2,...,)\in c_0$ and $t\in B(\mathbf{K},0,1)$. Then H_t is an isotopy, but H_t^{-1} is not jointly continuous at $(0,0)\in c_0\times B(\mathbf{K},0,1)$, since $B(\mathbf{K},0,1)=B(\mathbf{K},1,1)$. On the other hand, H_0^{-1} is continuous on c_0 and $H_t^{-1}(0)$ is continuous in $t\in B(\mathbf{K},0,1)$.

18. Definition. For a subset K of a totally disconnected Hausdorff topological space X let H_t be a 1-parameter family of homeomorphisms onto X, with $t\in B(\mathbf{K},0,1)$, so that $H_0=id$, $H_t(X)=X$ for each $t\ne 1$, $H_1(X\setminus K)=X$ and H_t and H_t^{-1} are jointly continuous in $(x,t)\in X\times B(\mathbf{K},0,1)$. Then H_t is called an invertible non-archimedean isotopy pushing K off X. Particularly, K may be a singleton $\{p\}$.

If $a\ne b\in B(\mathbf{K},0,1)$ with $0\le |a|\le 1$ and $0\le |1-b|<1$, we put $H[a,b]_t=id$ for each $t\in B(\mathbf{K},0,|a|)\setminus B(\mathbf{K},1,|1-b|)$, $H[a,b]_t=H_1$ for $t\in B(\mathbf{K},1,|1-b|)$, $H[a,b]_t=H_{(t-a)/(b-a)}$ for each $t\in B(\mathbf{K},0,1)\setminus[B(\mathbf{K},0,|a|)\cup B(\mathbf{K},1,|1-b|)]$. If $a,b,c\in B(\mathbf{K},0,1)$, $0\le|a|\le 1$, $0<|1-b|<1$, $0\le|1-c|\le|1-b|$, $a\ne b$, $b\ne c$, the composition $F[b,c]_t\circ H[a,b]_t$ of two isotopies is defined.

19. Lemma. *Let H^j be an invertible (non-archimedean) isotopy of a totally disconnected Hausdorff topological space X onto X for each $j\in\mathbf{N}$ and let $p_0\in X$ be a marked point. Let also*

(1) *for each point $x\in X\setminus\{p_0\}$ a neighborhood U of x in X and a natural number $n(U)\in\mathbf{N}$ exist so that $H_t^n=id$ on $H_1^{n(U)}\circ...\circ H_1^2\circ H_1^1(U)$ for each $n>n(U)$ and*

(2) *for each point $y\in X$ a neighborhood V of y in X and an integer $n(V)$ exist so that $(H_t^n)^{-1}=id$ on V and $H_1^{n(V)}\circ...\circ H_1^2\circ H_1^1(V)\subset(X\setminus\{p_0\})$ for each $n>n(V)$.*

Then $H_t=L\prod_{j=0}^{\infty}H^{j+1}[1-p^j,1-p^{j+1}]_t$ is an invertible (non-archimedean) isotopy pushing p_0 off X.

Proof. For each $t\in B(\mathbf{K},0,1)$, $|t|\ne 1$, a natural number $k\in\mathbf{N}$ exists such that $t\in B(\mathbf{K},0,p^{-k})$, consequently, $H^j[1-p^j,1-p^{j+1}]_t=id$ on X, when $j\ge 1$, since $|t|<|1-p^j|=1$. If $|t|=1$ and $t\ne 1$ a non-negative integer k exists such that $t\in B(\mathbf{K},1,p^{-k})\setminus B(\mathbf{K},1,p^{-k-1})$. Therefore, H_t

reduces to a finite product of homeomorphisms of X onto X. This implies that H_t and $(H_t)^{-1}$ are continuous for each $t \in B(\mathbf{K}, 0, 1)$, $t \neq 1$.

If $y \in X \setminus \{p_0\}$ and $t = 1$, the continuity of H at $(y, 1)$ follows from Condition 1. On the other hand, Condition 2 implies the continuity of H^{-1} at $(y, 1)$ for any $y \in X$.

20. Lemma. *Consider* $\mathbf{s}$ *as in §3. Let either* $X = \mathbf{s}$ *or* $X = c_0$. *Suppose that* T *is a compact subset in* X. *Then a homeomorphism* η *of* X *onto* X *exists so that* $\pi_1 \circ \eta(T)$ *is a single point in* $\mathbf{K}$, *where* $\pi_j : X \to \mathbf{K}_j$ *is a projection linear over* $\mathbf{K}$.

Proof. The topological vector spaces $\mathbf{s}$ and c_0 have projections π_j linear over $\mathbf{K}$. The demonstration below is given for $\mathbf{s}$, whilst that of c_0 is analogous. Take homeomorphisms f_j of $\mathbf{K}_1 \times \mathbf{K}_j$ onto itself, where $\mathbf{K}_j = \mathbf{K}$ for each $j \in \mathbf{N}$, satisfying two conditions:

(1) $f_j(x_1, x_j) = (x_1, y_j)$ for each $(x_1, x_j) \in \mathbf{K}_1 \times \mathbf{K}_j$, where $y_j \in \mathbf{K}_j$, and

(2) if D_j is a region in $\mathbf{K}_1 \times \mathbf{K}_j$ such that $D_j = \{(x, y) : \ (x, y) \in \mathbf{K}_1 \times \mathbf{K}_j, \ |x - x_0| \le a, \ |y - y_0| \le b\}$ for some $(x_0, y_0) \in \mathbf{K}_1 \times \mathbf{K}_j$, $a, b \in \Gamma_{\mathbf{K}}$, $\pi_{1,j}(T) \subset D_j$, where $\pi_{1,j} : \mathbf{s} \to \mathbf{K}_1 \times \mathbf{K}_j$ is a linear over $\mathbf{K}$ projection, and $f_j(D_j) = \{v : \ v \in \mathbf{K}_1 \times \mathbf{K}_j, \ v_1 = t_1(b_1 - a_1) + t_2(c_1 - a_1), \ v_2 = t_1(b_2 - a_2) + t_2(c_2 - a_2), \ t_1, t_2 \in B(\mathbf{K}, 0, 1)\}$, $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$ are marked points in $\mathbf{K}_1 \times \mathbf{K}_j$ and $v = (v_1, v_2)$ such that $f_j(D_j) \cap (w + (\mathbf{K}, 0)) = E_{w,j}$ with

$diam E_{w,j} := \sup_{x,y \in E_{w,j}} |x - y| \le p^{-j}$

for each $w \in \mathbf{K}_1 \times \mathbf{K}_j$.

Then a homeomorphism of f of $\mathbf{s}$ onto itself exists such that $f(x_1, x_2, ...) = (x_1, y_2, y_3, ...)$, where y_j is given by Condition (1) for each j. From the construction of f it follows that f is bijective and surjective. Since each f_j is continuous and $\mathbf{s}$ is supplied with the Tychonoff topology, then the mapping f is also continuous.

From Condition 2 it follows that $f(T) \cap (v + (\mathbf{K}, 0))$ is either a singleton or the void set for each $v \in \mathbf{s}$.

Put $\mathbf{s}_0 := \{x : \ x \in \mathbf{s}, \ x_1 = 0\}$ and let $\pi_0 : \mathbf{s} \to \mathbf{s}_0$ be the corresponding linear over $\mathbf{K}$ projection onto $\mathbf{s}_0$. Therefore, in accordance with the construction above $\pi_0|_{f(T)}$ is a homeomorphism from $f(T)$ into $\mathbf{s}_0$. Consider the restriction

$\Phi = \pi_1 \circ \pi_0^{-1}|_{\pi_0(f(T))}$. In view of Theorem 16 it has a continuous extension $\psi : \mathbf{s}_0 \to \mathbf{K}$, that is, $\psi|_{\pi_0(f(T))} = \Phi$.

There exists a homeomorphism ξ of $\mathbf{s}$ onto $\mathbf{s}$ so that $\xi|_{(p+(\mathbf{K},0))}(x) = x - \Phi(p)$ for each $p \in \mathbf{s}_0$ and $x \in p + (\mathbf{K}, 0) \subset \mathbf{s}$. The desired homeomorphism is $\eta = \xi \circ f$, since $\eta(T) \subset \mathbf{s}_0$ and $\pi_1(\eta(T)) = \{0\}$.

21. Theorem. *Let $\{T_j : \ j \in \mathbf{N}\}$ be a family of compact subsets of $X = \mathbf{s}$ or $X = c_0$. Then a homeomorphism of X onto X exists such that each subset $g(T_j)$ is infinitely deficient for each $j \in \mathbf{N}$.*

Proof. If β_j is a family of disjoint subsets of $\mathbf{N}$ so that $\bigcup_j \beta_j = \mathbf{N}$, then $\mathbf{s}$ and c_0 can be written as $\mathbf{s} = \prod_j \mathbf{s}^{\beta_j}$ (see §3), and $c_0 = c_0(c_0(\beta_j) : j)$ respectively, where

$c_0(Y_j : j) = \{y = (y_1, y_2, ...) : \ \forall j \ y_j \in Y_j, \forall \epsilon > 0$

$card\{j : \ \|y_j\| > \epsilon\} < \aleph_0; \ \|y\| = \sup_j \|y_j\|_{Y_j}\}$

for a family of Banach spaces Y_j over $\mathbf{K}$. Suppose that a mapping $\theta : \mathbf{N} \to \mathbf{N}$ has the property that $\theta^{-1}(j)$ is infinite for each $j \in \mathbf{N}$. Then each $\mathbf{s}^{\beta_j}$ or $c_0(\beta_j)$ respectively with $\beta_j = \theta^{-1}(j)$ is homeomorphic with $\mathbf{s}$ or c_0 respectively. In view of Lemma 20 a homeomorphism g_j of $\mathbf{s}^{\beta_j}$ or $c_0(\beta_j)$ onto itself so that $g_j(\pi_{\beta_j}(T_{\theta(j)}))$ is deficient relative to the first element of β_j. Put $g = \prod_j g_j$ for $\mathbf{s}$ or $g(y) = (g_1(y_1), g_2(y_2), ...)$ for each $y \in c_0 = c_0(c_0(\beta_j) : j)$ respectively. Then $g(T_j)$ is infinitely deficient for each $j \in \mathbf{N}$.

22. Remark. Let ${}_rH_t$ be a two-parameter (non-archimedean) family of homeomorphisms with $r \in B(\mathbf{K}, 0, 1) \setminus \{0\}$ and $t \in B(\mathbf{K}, 0, 1)$ such that for r fixed ${}_rH$ is an isotopy pushing the origin off $\mathbf{s}$. If ${}_rH_t x$ and $({}_rH_t)^{-1}x$ are continuous in r, t and $x \in \mathbf{s}$ or c_0, then the family $\{{}_rH_t : \ r \in B(\mathbf{K}, 0, 1) \setminus \{0\}$ and $t \in B(\mathbf{K}, 0, 1)\}$ is called an invertible continuous family of invertible (non-archimedean) isotopies. Henceforth, the case is considered, when ${}_rH_t$ is the identity outside the $|r|$-neighborhood of the origin in $\mathbf{s}$ or c_0 respectively. The topological space $\mathbf{s}$ is metrizable with the complete metric

(1) $d(x, y) = \sum_j \min(p^{-j}, |x_j - y_j|^{-1}) \in \mathcal{R}$.

23. Lemma. *There exists an invertible (non-archimedean) isotopy H pushing the origin x_0 off X, where $X = \mathbf{s}$.*

Proof. We consider an invertible (non-archimedean) isotopy F^j on $\mathbf{K}_1 \times \mathbf{K}_{j+1}$ satisfying the following conditions:

(1) $F^j_t(x_1, x_{j+1}) = (x_1, x_{j+1})$ for each $(x_1, x_{j+1}) \in \mathbf{K}_1 \times \mathbf{K}_{j+1}$ with $|x_1| < p^{j-1}$,

(2) F^j_1 maps the set $A_j := \{(z_1, z_2) : \ z_1 = \xi_1, z_2 = \xi_2\gamma + \xi_3(1-\gamma), \gamma \in B(\mathbf{K}, 0, 1)\}$ with $|\xi_1| = p^{j-1}$ and $|\xi_2| = p^{-j}$ and $|\xi_3| = p^j$ onto the set $B_j := \{(z_1, 0) : \ z_1 = (\xi_2 + \xi_4)\gamma + (\xi_3 + \xi_4)(1-\gamma), \ \gamma \in B(\mathbf{K}, 0, 1)\}$ with $|\xi_4| = p^{j+1}$ such that $F^1_j(\xi_1, \xi_2\gamma + \xi_3(1-\gamma)) = ((\xi_2 + \xi_4)\gamma + (\xi_3 + \xi_4)(1-\gamma), 0)$ for each γ, where $\xi_1, ..., \xi_4 \in \mathbf{K}$. Therefore, one gets $\xi_2\gamma + \xi_3(1-\gamma) = 0 \Leftrightarrow (\xi_2 - \xi_3)\gamma = -\xi_3, \ \ \gamma = \frac{\xi_3}{\xi_3 - \xi_2}$ hence $|\gamma| = 1$ and $\gamma \in B(\mathbf{K}, 0, 1)$ and there exists $z \in A_j$ with $|z_1| = |\xi_1| = p^{j-1}$ and $z_2 = 0$. On the other hand, if $z \in B_j$, then $|z_1| = p^{j+1}$. Conditions $(1, 2)$ can be satisfied using a partition of X into a disjoint union of clopen subsets.

We consider points $a \in \mathbf{K}_2 \times ... \times \mathbf{K}_j$ and $b \in \mathbf{K}_1 \times \mathbf{K}_{j+1}$ and $c = (c_1, c_2, ...) \in X$ such that $c_1 = 0$, ...,$c_j = 0$ and put $\phi_j(a) = \xi \in \mathbf{K}$ when $j \geq 2$ such that $|\xi| = \max(0; q \in \Gamma_{\mathbf{K}}, \ q \leq 1 - p^{j+1}\|a\|\}$, where $\|a\| = \sup_{2 \leq l \leq j} |a_l|$, $a = (a_2, ..., a_j)$, $a_l \in \mathbf{K}_l$ for each l. Put also $\phi_1(a) = 1$. Then a non-archimedean isotopy H^j of X onto X exists so that $H^j_t(a, b, c) = (a, F^j_{t\phi_j(a)}(b), c)$. The function $\phi_j(a)$ is continuous, since balls $B(\mathbf{K}_2 \times ... \times \mathbf{K}_j, x, r)$ are clopen in the normed space $\mathbf{K}_2 \times ... \times \mathbf{K}_j$ for each $x \in \mathbf{K}_2 \times ... \times \mathbf{K}_j$ and $r \in \Gamma_{\mathbf{K}}$.

The constructed sequence H^j of non-archimedean isotopies satisfies Conditions 1 and 2 of Lemma 19, hence

$H^1_t = L \prod_{j=0}^{\infty} H^{j+1}[1 - p^j, 1 - p^{j+1}]_t$

is an invertible non-archimedean isotopy pushing x_0 off X.

It remains to verify that $\{H^j\}$ satisfies Conditions $19(1, 2)$. For this purpose take an arbitrary point $x \in X \setminus \{x_0\}$ with $x = (0, ..., 0, a_j, a_{j+1}, ...)$, where a_j is the first non zero coordinate of x. For the composition $Q^j := H^j_1 \circ ... H^2_1 \circ H^1_1$ let $Q^j(x) = (b_1, ..., b_j, b_{j+1}, a_{j+2}, ...)$ and $Q^{j+1}(x) = (c_1, b_2, ..., b_{j+1} c_{j+2}, a_{j+3}, ...)$, where $b_2 = 0$,...,$b_j = 0$ for each $j \geq 2$. A neighborhood U of x and an integer k exist, when one of the coordinates $b_2, ..., b_{j+1}, c_{j+2}$ is non zero, so that $\phi_k = 0$ on $\pi_{(2,...,k)} Q^k(U)$, where $\pi_{(l,...,k)} : X \to \mathbf{K}_l \times ... \times \mathbf{K}_k$ is a $\mathbf{K}$-linear projection with $l < k$. Therefore, $H^v_t = id$ on $Q^k(U)$ for $v \geq k$. Particularly, $Q^j(0, ..., 0, a_{j+2}, ...) = (\xi_4, 0, ..., 0, a_{j+2}, ...)$, where $|\xi_4| = p^{j+1}$. If $|b_1| \neq p^j$ and $c_{j+2} = 0$, then $|c_1| < p^{j+2}$ due to Condition (2). In the case $|c_1| < p^{j+2}$ a

neighborhood U of x exists so that $|\pi_1 \circ Q^{j+1}(y)| < p^{j+2}$ for each $y \in U$. Thus $H^k_t = id$ on $Q^{j+1}(U)$ for each $k \ge j+1$. That is, Condition 19(1) is fulfilled. Then Condition 19(2) is satisfied, since $H^{j+1}_t(y) = y$ when $|y_1| < p^j$, while $|\pi_1 \circ Q^j(x_0)| = p^{j+1}$.

24. Lemma. *Let $X = X_1 \times X_2$, where either $X_1 = \mathbf{s}^\alpha$ and $X_2 = \mathbf{s}^\beta$ or $X_1 = c_0(\alpha, \mathbf{K})$ and $X_2 = c_0(\beta, \mathbf{K})$ with $card(\alpha) = card(\beta) = \aleph_0$, X_1 and X_2 have origins $x_{0,1}$ and $x_{0,2}$ respectively. Suppose that H is an invertible (non-archimedean) isotopy pushing $x_{0,1}$ off X_1 and $\phi_r : X_2 \to B(\mathbf{K}, 0, 1)$ is a continuous one parameter family of maps with $r \in B(\mathbf{K}, 0, 1) \setminus \{0\}$ and $\phi_r^{-1}(1) = x_{0,2}$ and $w : B(\mathbf{K}, 0, 1) \times B(\mathbf{K}, 0, 1) \to B(\mathbf{K}, 0, 1)$ is continuous such that $w : B(\mathbf{K}, 0, 1) \setminus \{0\} \times B(\mathbf{K}, 0, 1) \setminus \{0\} \to B(\mathbf{K}, 0, 1) \setminus \{0\}$ is a mapping onto $B(\mathbf{K}, 0, 1) \setminus \{0\}$ so that $w^{-1}(1) = (1, 1)$ and $w([\{0\} \times B(\mathbf{K}, 0, 1)] \cup [B(\mathbf{K}, 0, 1) \times \{0\}]) = \{0\}$. Then ${}_rH_t(x, y) = (H_{w(t,\phi_r(y))}(x), y)$ defines an invertible continuous one parameter with $r \in B(\mathbf{K}, 0, 1)$ family of invertible (non-archimedean) isotopies pushing the origin off X for each r.*

Proof. In two considered cases X is either $\mathbf{s}$ or c_0. Since $\phi_r(y)$ is continuous on $(B(\mathbf{K}, 0, 1) \setminus \{0\}) \times X_2$, then the composite mapping $w(t, \phi_r(y)) : B(\mathbf{K}, 0, 1) \times [B(\mathbf{K}, 0, 1) \setminus \{0\}] \times X_2 \to B(\mathbf{K}, 0, 1)$ is continuous. But H_b^{-1} is a continuous (non-archimedean) isotopy. Therefore, ${}_rH_t^{-1}(x, y) = (H^{-1}_{w(t,\phi_r(y))}(x), y)$ is the continuous non-arhimedean isotopy. On the other hand, $\phi_r(x_{0,2}) = 1$ for each r, consequently, ${}_rH_1(x_{0,1}, x_{0,2}) = (H_1(x_{0,1}), x_{0,2}) \ne x_0$ and ${}_rH_1(X) = \{H_{w(t,\phi_r(y))}(X_1), y) : y \in X_2\} = X$ for each r. Moreover, ${}_rH_0(x, y) = (H_{w(0,\phi_r(y))}(x, y) = (H_0(x), y) = (x, y) = id(x, y)$, since $w(0, b) = 0$ for each $b \in B(\mathbf{K}, 0, 1)$.

25. Lemma. *An invertible continuous non-archimedean one parameter family of invertible isotopies ${}_r\tilde{H}$ exists with $r \in B(\mathbf{K}, 0, 1) \setminus \{0\}$ each pushing the origin x_0 off $\mathbf{s}$ so that ${}_r\tilde{H}_t$ is the identity mapping outside a neighbourhood U_r of x_0.*

Proof. Let α and β be two infinite subsets in $\mathbf{N}$ such that $\beta = \mathbf{N} \setminus \alpha$. We define the metric on $X^\gamma = \mathbf{s}^\gamma$ by the formula

$$d_\gamma(x, y) = \sum_{j \in \gamma} \min(p^{-j}, |x_j - y_j|) \in \mathcal{R},$$

where $x, y \in \mathbf{s}^\gamma$. Take in particular $\gamma = \alpha$ or $\gamma = \beta$. Suppose wihout loss

of generality that $1, 2, 3 \in \beta$, hence the diameter of $\mathbf{s}^\alpha$ is less than p^{-3}, $diam(\mathbf{s}^\alpha) := \sup_{x,y \in \mathbf{s}^\alpha} d_\alpha(x, y) < p^{-3}$.

Next we consider a non-archimedean isotopy H pushing the origin off $\mathbf{s}^\alpha$ (see Lemma 23). There exists a continuous map ϕ of $\mathbf{s}^\beta$ on $B(\mathbf{K}, 0, 1)$ so that $\phi^{-1}(1) = x_{0,2}$ while $\phi(x)$ is zero for each $x \in \mathbf{s}^\beta \setminus B(\mathbf{s}^\beta, x_{0,2}, p^{-3})$, where $B(\mathbf{s}^\beta, x_{0,2}, q) = \{y \in \mathbf{s}^\beta : \ d_\beta(x, y) \le q\}$, $q \in \Gamma_{\mathbf{K}}$. This is possible, since $B(\mathbf{s}^\beta, x_{0,2}, q)$ is clopen in $\mathbf{s}^\beta$ for $q > 0$. Put ${}_1H_t(x, y) = (H_{w(t,\phi(y))}(x), y)$ when $d((x, y), x_0) \ge p^{-2}$, since $1, 2, 3 \in \beta$ and $d_\beta(y, x_{0,2}) \ge p^{-3}$ and hence $\phi(y) = 0$. We denote by l the least natural number in α and by k a natural number in β greater than l. Let β_1 be the subset of $\mathbf{N}$ formed from β by the substitution $k \mapsto l$, while α_1 is made from α substituting l with k.

There exists a family F_λ, with $\lambda \in B(\mathbf{K}, 0, 1)$, of transfromations of $\mathbf{s}$ given by the formula: $F_\lambda(x_1, x_2, ...) = (y_1, y_2, ...)$ with $y_j = x_j$ for each $j \in \mathbf{N} \setminus \{l, k\}$; whilst $y_l = (1 - \lambda)x_l$ when $|(1 - \lambda)x_l| > |\lambda x_k|$, $y_l = \lambda x_k$ when $|\lambda x_k| \ge |(1 - \lambda)x_l|$; $y_k = \lambda x_l$ when $|\lambda x_l| \ge |(1 - \lambda)x_k|$, $y_k = (1 - \lambda)x_k$ when $|(1 - \lambda)x_k| > |\lambda x_l|$.

Let $f(r)$ be a locally affine continuous mapping from $B(\mathbf{K}, 0, 1)$ onto $B(\mathbf{K}, 0, 1)$ so that $f(B(\mathbf{K}, \xi_1, p^{-2})) = B(\mathbf{K}, 1, p^{-2})$ for $\xi_1 \in \mathbf{K}$ with $|\xi_1| = p^{-1}$, $f(B(\mathbf{K}, 1, p^{-l-1}) \setminus B(\mathbf{K}, 1, p^{-l-2})) = B(\mathbf{K}, 0, p^{-l-1}) \setminus B(\mathbf{K}, 0, p^{-l-2})$ for each natural number l, $f(1) = 0$.

Define the mapping ${}_rH_t := F^{-1}_{f(r)} \circ {}_1H_t \circ F_{f(r)}$ for each $r \in W := [B(\mathbf{K}, \xi_1, p^{-2}) \cup B(\mathbf{K}, 1, p^{-2})]$. If $d(x, x_0) \ge p^{-2}$ and $r \in W$, then $d(F_{f(r)}(x), x_0) \ge d(x, x_0)$ and hence ${}_rH_t(x) = x$, since $|1 - \lambda| = 1$ for any $\lambda \in B(\mathbf{K}, 0, p^{-1})$ and $|\lambda| = 1$ for each $\lambda \in B(\mathbf{K}, 1, p^{-1})$.

Now we define ${}_rH$ for $r \in B(\mathbf{K}, 1, p^{-1}) \setminus W =: C_1$ so that ${}_rH_t(x, y) = (S_{w(t,\phi_r(y))}(x), y)$ for each $x \in \mathbf{s}^{\alpha_1}$ and $y \in \mathbf{s}^{\beta_1}$, where T is a linear over $\mathbf{K}$ operator on $\mathbf{s}$ interchanging a finite number of coordinates, $S_t := T^{-1} \circ H_t \circ T$.

Let ξ_1 be a marked point as above and define ϕ_{ξ_1} as a map of $\mathbf{s}^{\beta_1}$ onto $B(\mathbf{K}, 0, 1)$ such that $\phi^{-1}_{\xi_1}(1) = x_{0,2}$ and $\phi_{\xi_1}(y) = 0$ for each $y \in \mathbf{s}^{\beta_1} \setminus U_1$, where U_1 is a small neighborhood of $x_{0,2}$, $U_1 = \{y \in \mathbf{s}^{\beta_1} : \ d_{\beta_1}(x_{0,2}, y) \le p^{-3}\}$. This mapping ϕ_{ξ_1} can be presented as the composition $\phi_{\xi_1} = \phi \circ T_1$, where a mapping $T_1 : \mathbf{s}^{\beta_1} \to \mathbf{s}^{\beta_1}$ is given by the formula $T_1(x_1, x_2, ..) = (y_1, y_2, ...)$ with $y_j = x_j$ for each $j \notin \{l, k\}$, also $y_l = x_k$ and $y_k = -x_l$.

If $d(z,x_0) \geq p^{-2}$, $z=(x,y)$, $x \in \mathbf{s}^{\alpha_1}$, $y \in \mathbf{s}^{\beta_1}$, then $\phi_{\xi_1}(y)=0$, since $d(y,x_{0,2}) \geq p^{-2}$ and $d(T_1(y),x_{0,2}) \geq p^{-2}$. For $\xi_v \in B(\mathbf{K},0,p^{-v}) \setminus B(\mathbf{K},0,p^{-v-1}) =: C_v$ with a natural number $2 \leq v \in \mathbf{N}$, let ϕ_v be a continuous mapping of $\mathbf{s}^{\beta_1}$ onto $B(\mathbf{K},0,1)$ so that $|\phi_{\xi_{v+1}}(y)| \leq |\phi_{\xi_v}(y)|$ for each $y \in \mathbf{s}^{\beta_1}$, also $\phi_{\xi_v}^{-1}(1)=x_{0,2}$, $\phi_{\xi_v}=0$ for each y with $d(y,x_{0,v})>p^{-v-2}$, where $x_{0,v}$ is the origin in $\mathbf{s}^{\beta_v}$, T_v and α_v and β_v are defined by induction. For each $y \in \mathbf{s}^{\beta_v}$ let

$$\frac{\phi_{\xi_v}(y)-\phi_r(y)}{\phi_{\xi_{v-1}}(y)-\phi_r(y)}=\frac{\xi_v-r}{\xi_{v-1}-r}$$

when $\phi_{\xi_v}(y) \neq \phi_{\xi_{v-1}}(y)$ for each $r \in C_v \setminus \{\xi_v\}$, while $\phi_r(y)=\phi_{\xi_v}(y)$ when $\phi_{\xi_v}(y)=\phi_{\xi_{v-1}}(y)$. Put ${}_rH_t(x,y)=(S_{w(t,\phi_r(y))}(x),y)$ for each $r \in C_v$ by induction on $2 \leq v \in \mathbf{N}$.

On the other hand, due to Lemmas 6 and 15, Theorem 16 there exists a homeomorphism θ of topological spaces from $B(\mathbf{K},0,1)\setminus\{0\}$ onto $[B(\mathbf{K},\xi_1,p^{-2}) \cup B(\mathbf{K},0,p^{-2}) \cup B(\mathbf{K},1,p^{-2})] \setminus \{0\}$ such that $\theta(\xi_1)=\xi_1$, $\theta(1)=1$ and $\lim_{t\to 0}\theta(t)=0$, hence ${}_{\theta(r)}H_t$ is the claimed isotopy ${}_r\tilde{H}_t$.

26. Lemma. *There exists an invertible non-archimedean isotopy F pushing a point off A_0.*

Proof. We consider the set $c_0(1) := \{x \in c_0 : x_1=1, x=(x_1,x_2,...), \forall j \in \mathbf{N}\ x_j \in \mathbf{K}\}$ and points $q_j=(q_{j,1},...,q_{j,j},0,0,..) \in c_0$, $q_{j,1}=1$,...,$q_{j,j}=1$ for each $j \in \mathbf{N}$. Take the neighborhood $U_j=\{(1,x_2,...) \in c_0(1) : \max_{i=2}^j |1-x_i|<p^{-j}\}$ of the point q_j. Define $H^1_t(x)=x+te_2$, $H^i_t(x)=x+p^{1-i}t\eta_i(x)e_{i+1}$, where $\eta_i(x) \in \mathbf{K}$, $|\eta_i(x)|=d(x,c_0(1)\setminus U_i)$ for each $i \geq 2$, and

$H_t=L\prod_{i=0}^\infty H^{i+1}[1-p^i,1-p^{i+1}]_t$,

where $e_i=(0,...,0,1,0,...)$ is the vector with unit i-th coordinate and zero others. In view of Theorem 4.1 [1] and Lemma 19 H is an invertible non-archimedean isotopy pushing q_1 out of $c_0(1)$.

The topological space $c_0(1)$ is the closed subset in c_0 and is a union of $\omega(\mathbf{K})|\Gamma_{\mathbf{K}}|\aleph_0$ disjoint balls $B(c_0(1),x,r)$, where $x \in c_0(1)$, $r \in \Gamma_{\mathbf{K}}$, $\omega(P)$ denotes the topological weight of a topological space P, whilst $card(S)=|S|$ denotes the cardinality of a set S. Then the topological space A_0 is also the closed subset in c_0. The topological spaces c_0 and $c_0 \oplus \mathbf{K}$ are linearly

homeomorphic, but $c_0 \oplus \mathbf{K}$ is also linearly homeomorphic with c, consequently, c_0 and c are linearly homeomorphic. Consider the homemorphism $\nu : c_0 \to c$ of c_0 onto c.

Particularly, consider linear topological isomorphism $\nu : c_0 \to c$ such that $\nu(x) = y$ with $y_k = \sum_{j=1}^k x_j$ for each $k \in \mathbf{N}$, since the series $\sum_{j=1}^k x_j$ converges if and only if $\lim_j x_j = 0$ due to the ultrametric inequality. Moreover, the topological space $\nu(A_0) =: W_0$ is the disjoint union of $\omega(\mathbf{K})|\Gamma_{\mathbf{K}}|\aleph_0$ balls $B(c, y, r)$ with $y \in W_0$ and $\Gamma_{\mathbf{K}} \ni r \le p^{-1}$, since from $y \in W_0$ and $z \in B(c, 0, r)$ it follows $|y_k + z_k| \le \max(|y_k|, |z_k|) \le 1$ for each natural number k and from $|y_l| = 1$ it follows $|y_l + z_l| = 1$, while $\nu(A_0) = \{y \in c : \sup_k |y_k| = 1\}$.

Therefore, each two balls $B(c_0(1), x, r)$ and $B(c, y, q)$ with $r, q \in \Gamma_{\mathbf{K}}$ are homeomorphic. Thus A_0 and $c_0(1)$ are homeomorphic. Therefore, there exists an invertible non-archimedean isotopy $F_t(y) = \psi^{-1} \circ H_t \circ \psi(y)$, where ψ is a homeomorphism of A_0 onto $c_0(1)$ described above.

27. Corollary. *There exists an invertible non-archimedean isotopy G pushing a point off c_0.*

Proof. The topological spaces c_0 and $c_0(1)$ are homeomorphic with a homeomorphism $\eta : c_0 \to c_0(1)$. Indeed, the topological space c_0 can be presented as the disjoint union of $\omega(\mathbf{K})|\Gamma_{\mathbf{K}}|\aleph_0$ balls $B(c_0, x, r)$ with $x_0 \in c_0$ and $r \in \Gamma_{\mathbf{K}}$. From §26 it follows that $G_t(y) = \eta^{-1} \circ H_t \circ \eta(y)$ is an invertible non-archimedean isotopy pushing a point off c_0.

28. Lemma. *There exists an invertible non-archimedean isotopy H pushing the origin $x_0 = 0$ off c_0 so that H_t is the identity outside the unit ball in c_0 containing zero for each $t \in B(\mathbf{K}, 0, 1)$.*

Proof. The unit ball $B(c_0, 0, 1)$ is clopen in c_0. On the other hand, $B(c_0, 0, 1)$ is homeomorphic with A_0. An invertible non-archimedean isotopy evidently has an extension from $B(c_0, 0, 1)$ onto c_0. From Lemma 26 and the equality $B(c_0, 0, 1) = B(c_0, x, 1)$ for each $x \in B(c_0, 0, 1)$, particularly, for $\|x\| = 1$, the statement of this lemma follows.

29. Lemma. *There exists an invertible continuous one parameter $r \in B(\mathbf{K}, 0, 1)$ family of invertible non-archimedean isotopies ${}_rH$ each pushing the origin x_0 off c_0 such that ${}_rH_t$ is the identity outside $B(c_0, 0, |r|)$ for each $t \in B(\mathbf{K}, 0, 1)$.*

Proof. If $r \in \mathbf{K} \setminus \{0\}$, then $m_r : c_0 \to c_0$ is the linear homeomorphism of c_0 onto c_0, where $m_r(x) = rx$ for any $x \in c_0$. Therefore, the desired isotopy is given by the formula: ${}_rH_t = m_r \circ H_t \circ m_{1/r}$, where H_t is the isotopy provided by Lemma 28.

30. Lemma. *Let X be one of the topological spaces c_0 or $\mathbf{s}$, let also Ω be an open covering of X. Suppose that α is an infinite proper subset of $\mathbf{N}$ and Q is a closed subset in X deficient with respect to α. Then for any open subset U containing Q a homeomorphism g of $X \setminus Q$ onto X exists such that g is limited by Ω, $h|_{X \setminus U} = id$, $\pi_j(x) = \pi_j(g(x))$ for each $j \in \beta$, where $\beta = \mathbf{N} \setminus \alpha$.*

Proof. In view of Lemma 4 there exists the decomposition of X into the product of two topological spaces $X = X^\alpha \times X^\beta$, where either $X^\alpha = c_0(\alpha, \mathbf{K})$ or $X^\alpha = \mathbf{s}^\alpha$ for either $X = c_0$ or $X = \mathbf{s}$ respectively.

Without loss of generality we consider the case $\pi_\alpha(0) = x_{0,\alpha}$, where $x_{0,\alpha} = 0 \in X^\alpha$. Put $\pi_\beta(Q) = Q'$, hence $Q' \subset X^\beta$.

Take an open covering $\mathcal{W}$ of X of mesh less than one relative to the norm on c_0 or the metric d on $\mathbf{s}$ so that

(1) $\mathcal{W}$ is a refinement of Ω,

(2) if $S \in \mathcal{W}$ and $S \cap Q \neq \emptyset$, then $S \subset U$.

Then let $v(y)$ be a function defined by the formula:

(3) $v(y) = \sup\{\epsilon : \ \epsilon \in \Gamma_{\mathbf{K}}, \ \exists \ S \in \mathcal{W} \ B(X, (x_{0,\alpha}, y), p\epsilon) \subset S\}$, where $y \in X^\beta$. This function satisfies the inequality $v(y) > 0$ for each $y \in X^\beta$.

For a set $A = \{y \in X^\beta : \ B(X, (x_{0,\alpha}, y), v(y)) \text{ is not contained in } U\}$ we define the function

(4) $\tau(y) := d(y, A)/(d(y, A) + d(y, Q'))$, when A is non void, $A \neq \emptyset$, while $\tau(y) = 1$ for $A = \emptyset$. This definition implies that $(cl\ A) \cap Q' = \emptyset$, $\tau(y) = 1$ for each $y \in Q'$ and $\tau(y) = 0$ for any $y \in A$.

In accordance with Lemmas 25 and 29 an invertibly continuous family ${}_rH$ with $r \in B(\mathbf{K}, 0, 1) \setminus \{0\}$ of invertible non-archimedean isotopies exist each pushing $x_{0,\alpha}$ off X^α and so that ${}_rH_t|_{X^\alpha \setminus B(X^\alpha, x_{0,\alpha}, |r|)} = id$.

We then define the mapping

(5) $g(x, y) = ({}_{w(y)}H_{\mu(y)}(x), y)$ and verify below that it satisfies the desired properties, where $w(y)$ and $\mu(y)$ are continuous mappings from X^β into

$\mathbf{K}$ such that $\frac{v(y)}{p} \le |w(y)| \le v(y)$ and $\frac{\tau(y)}{p} \le |\mu(y)| \le \tau(y)$ for each $y \in X^\beta$, where $\mu(y) = 1$ for each $y \in Q'$. This implies that ${}_{w(y)}H_{\mu(y)}$ is a homeomorphism of $(X^\alpha \setminus \{x_{0,\alpha}\}) \times \{y\}$ onto $X^\alpha \times \{y\}$, when $y \in Q'$, since $\mu(y) = 1$ for each $y \in Q'$. Then ${}_{w(y)}H_{\mu(y)}$ is a homeomorphism of $X^\alpha \times \{y\}$ onto $X^\alpha \times \{y\}$, since $\tau(y) < 1$ for any $y \in X^\beta \setminus Q'$.

As the composition of continuous mappings, the mapping $g(x, y)$ given by Formula (5) also is continuous, since ${}_rH$ is a continuous family of isotopies. The inverse mapping is given by the formula $g^{-1}(x, y) = ({}_{w(y)}H^{-1}_{\mu(y)}(x), y)$. Since ${}_rH$ is an invertibly continuous family of invertible non-archimedean isotopies, then g^{-1} is continuous. Thus g is a homeomorphism of $X \setminus Q$ onto X.

Formula (5) implies that $\pi_j(z) = \pi_j(g(z))$ for every $z \in X$ and $j \in \beta$, since $z = (x, y)$ with $x \in X^\alpha$ and $y \in X^\beta$.

On the other hand, the mapping g is the identity on $X \setminus U$, since

(6) ${}_rH_t|_{X^\alpha \setminus B(X^\alpha, x_{0,\alpha}, |r|)} = id$ and $B(X, (x_{0,\alpha}, y), |w(y)|) \subset U$ when $\tau(y) \ne 0$. Moreover, Condition (6) implies that either $g(x, y) = (x, y)$ or x and ${}_{w(y)}H^{-1}_{\mu(y)}(x) \in B(X^\alpha, x_{0,\alpha}, |w(y)|)$. But the definition of $w(y)$ means that (x, y) and $g(x, y)$ belong to an element S of a covering ν containing (x_0, y), i.e. $(x_0, y) \in S$.

31. Theorem. *Let X be either c_0 or $\mathbf{s}$, let also U be an open subset in X. Suppose that $\{K_j :\ j \in \mathbf{N}\}$ is a sequence of closed subsets of X so that $K_j \subset U$ and K_j has an infinite deficiency for each j. Then a homeomorphism g of $X \setminus \bigcup_{j=1}^\infty K_j$ onto X exists so that $g|_{X\setminus U} = id$.*

Proof. The definition of the infinite deficiency implies that for each K_j an infinite subset $\beta_j \subset \mathbf{N}$ exists so that K_j is deficient with respect to β_j. The family $\{K_j : j \in \mathbf{N}\}$ is countable and $\pi_l(K_j)$ consists of a single element for each $l \in \beta_j$, consequently, there exists a disjoint family of infinite subsets $\alpha_j \subset \beta_j$, $\alpha_i \cap \alpha_j = \emptyset$ for each $i \ne j$, such that K_j is infinite deficient with respect to α_j.

In accordance with Lemma 30 a sequence of homeomorphisms g_j and a sequence of coverings G_j with $j \in \mathbf{N}$ satisfying conditions of Theorem 4.3 [1] exist. Mention that results of §4 [1] are also valid for a metric space with a metric d satisfying the non-archimedean inequality and with values

in $\mathcal{R}$ substituting in proofs $1/2$ on $1/p < 1 \in \mathcal{R}$, since a ring $\mathcal{R}$ is complete as a uniform space. That is, for each natural number $j \ge 1$ there exists a homeomorphism g_j of $X \setminus g_{j-1} \circ ... \circ g_1 \circ id(K_j \setminus \bigcup_{l=1}^{j-1} K_l)$ onto X with $g_0 = id$ such that $g_j(x) = y$ for each $x \in X \setminus g_{j-1} \circ ... \circ g_1 \circ id(K_j \setminus \bigcup_{l=1}^{j-1} K_l)$ with $x_k = y_k$ for any $k \in \mathbf{N} \setminus \alpha_j$. Therefore, the mapping $L \prod_{j=1}^{\infty} g_j$ is a homeomorphism of $X \setminus \bigcup_{j=1}^{\infty} K_j$ onto X.

32. Corollary. *The topological space $c_0 \setminus \bigcup_{j=1}^{\infty} E^j$ is homeomorphic with c_0.*

33. Corollary. *If $\{C_j : j \in \mathbf{N}\}$ is a countable family of compact subsets of $\mathbf{s}$, then $\mathbf{s} \setminus \bigcup_{j=1}^{\infty} C_j$ is homeomorphic to $\mathbf{s}$.*

Proof. A homeomorphism g of $\mathbf{s}$ onto itself so that $g(C_j)$ is infinitely deficient for each $j \in \mathbf{N}$ exists due to Theorem 21. Then from Theorem 31 the assertion of this corollary follows.

34. Corollary. *Let U be an open subset of c_0, let also K_j be a compact subset in U for each natural number j. Then a homeomorphism g of $c_0 \setminus \bigcup_{j=1}^{\infty} K_j$ onto c_0 exists so that $g|_{c_0 \setminus U} = id$.*

Proof. In view of Lemma 20 and Theorem 21 a homeomorphism η of c_0 onto c_0 exists such that $\eta(K_j)$ is infinitely deficient. Then by Theorem 31 there exists a homeomorphism ξ of $c_0 \setminus \bigcup_{j=1}^{\infty} g(K_j)$ onto c_0 so that $\xi|_{c_0 \setminus \eta(U)} = id$, consequently, $g = \eta^{-1} \circ \xi \circ \eta$.

35. Lemma. *There exists a homeomorphism of $c_0 \setminus \bigcup_{n=1}^{\infty} E^n$ with A_2^*.*

Proof. The topological space $A_2 := \{x \in c_0 : \sup_{1 \le j \le k \in \mathbf{N}} |\sum_{j=1}^{k} x_j| = 1,\ \sum_{j=1}^{k} x_j \ne 1\ \forall k \in \mathbf{N},\ \sum_{j=1}^{\infty} x_j = 1\}$ is homeomorphic with $A_3 := \{y \in c : \sup_{k \in \mathbf{N}} |y_k| = 1, y_k \ne 0 \forall k\}$. Indeed, the mapping $h_1(x) = y$ such that $y_k = (\sum_{j=1}^{k} x_j)$ is the required homeomorphism, since $x_1 = 1 - \sum_{j=2}^{\infty} x_j$ and $|x_1| \le |1|$, $\lim_k y_k = (\sum_{j=1}^{\infty} x_j) = 1$. Therefore, the space A_2^* is homeomorphic with $A_3^* = A_3 \setminus \bigcup_n E^n$, since $x_n = 0$ is equivalent to $y_n = y_{n-1}$ and $E^n = E^1 \oplus E^{n-1}$, where E^n is homeomorphic with $\mathbf{K}^n$.

The topological spaces c and A_3 are homeomorphic as disjoint unions of $|\Gamma_{\mathbf{K}}| \aleph_0 \omega(\mathbf{K})$ balls $B(c, x, q)$ with $q \le 1/p$, $q \in \Gamma_{\mathbf{K}}$, where $x \in c$ or $x \in A_3$ respectively. On the other hand, c_0 and c are homeomorphic (see §26), whilst the topological spaces c_0 and $c_0 \setminus \bigcup_{j=1}^{\infty} E^j$ are homeomorphic by Corollary 32. Thus $c_0 \setminus \bigcup_{j=1}^{\infty} E^j$ is homeomorphic with A_3^* due to Theorem 31 and hence

with A_2^*.

36. Lemma. *The topological spaces s_* and s are homeomorphic.*

Proof. Consider the topological space s realized as in §7. The mapping $h(x) = y$ with $y_k = x_k - 1$ for each $k \in \mathbf{N}$ is the isomorphism of s with

$$s_1 := \prod_{j=1}^{\infty} (B(\mathbf{K}, 0, 1)_j \setminus \{0\}).$$

Put $C_1^j := \{y \in P : \ y_k = -1 \text{ or } y_k = 0 \ \forall k > j\}$, where

$$P = \prod_{j=1}^{\infty} B(\mathbf{K}, 0, 1)_j.$$

Since a field $\mathbf{K}$ is locally compact and the unit ball $B(\mathbf{K}, 0, 1)$ of it is compact, then the topological space P is compact and each subset C_1^j in it is compact. In view of Theorem 4.3 [1], Remark 7 and Lemma 25 and Corollary 33 the topological spaces s_1 and $s_{1*} := s_1 \setminus \bigcup_{j=1}^{\infty} C_1^j$ are homeomorphic. Therefore, the topological spaces s_* and s are homeomorphic.

37. Remark. Thus Theorem 2 follows from the sequence of homeomorphisms $c_0 \approx c_0 \setminus \bigcup_j E^j \approx A_2^* \approx A_1^* \approx s_* \approx s$ demonstrated above in Corollary 32, Lemmas 35, 12, 10 and 36.

Theorem 2 has a generalization over a field $\mathbf{F}$ of separable type over $\mathbf{K}$, i.e. when $\mathbf{F}$ has a multiplicative norm extending that of $\mathbf{K}$ such that $\Gamma_{\mathbf{K}} \subset \Gamma_{\mathbf{F}}$ and $\mathbf{F}$ has an equivalent norm $| * |_{1,\mathbf{F}}$ not necessarily multiplicative so that $|a|_{\mathbf{F}}/p \le |a|_{1,\mathbf{F}} \le p|a|_{\mathbf{F}}$ and $|a|_{1,\mathbf{F}} \in \Gamma_{\mathbf{K}} \bigcup \{0\}$ for each $a \in \mathbf{F}$ and $(\mathbf{F}, | * |_{1,\mathbf{F}})$ is isomorphic as the Banach space over $\mathbf{K}$ with $c_0(\omega_0, \mathbf{K})$. Indeed, $c_0(\omega_0, c_0(\omega_0, \mathbf{K}))$ is isomorphic with $c_0(\omega_0, \mathbf{F})$ and $(\mathbf{K}^{\omega_0})^{\omega_0}$ is isomorphic with $(c_0(\omega_0, \mathbf{K}))^{\omega_0}$ and hence with $\mathbf{F}^{\omega_0}$, consequently, the topological spaces $c_0(\omega_0, \mathbf{F})$ and $\mathbf{F}^{\omega_0}$ are homeomorphic.

38. Corollary. *If the topological $\mathbf{K}$ linear space $c_0(\alpha)$ with $card(\alpha) > \aleph_0$ is supplied with the projective limit topology τ_{pr} induced by $\mathbf{K}$ linear projection operators $\pi^\alpha_\beta : c_0(\alpha) \to c_0(\beta)$ associated with the standard base $\{e_j : \ j \in \alpha\}$ for each $\beta \subset \alpha$ with $card(\beta) = \aleph_0$, then $(c_0(\alpha), \tau_{pr})$ is topologically homeomorphic with $(\mathbf{K}^\alpha, \tau_{ty})$.*

Proof. Mention the fact that the topological space $(\mathbf{K}^\alpha, \tau_{ty})$ can be presented as the projective limit of topological spaces $(\mathbf{K}^\beta, \tau_{ty})$ with $\beta \subset \alpha$,

$card(\beta) = \aleph_0$, when $card(\alpha) > \aleph_0$ (see also [3]). Therefore, applying Theorem 2 we get this corollary.

39. Conclusion. The results of this paper can be used for subsequent studies of non-archimedean topological vector spaces and manifolds on them.

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